motion, the lower equilibrium position.

We now consider the upper equilibrium positon of the pendulum, which becomes stable for $k^2 < 0,5$ /10/. Taking the average of the force function and the kinetic energy just as we did for the lower equilibrium position, we again obtain the maximum of the mean $U$ and the minimum of the mean $T$ at the stable upper equilibrium position.

Therefore, an approximate analysis of the values of $\langle U\rangle, \langle T\rangle$ in specific mechanics problems verifies the V.V. Beletskii hypothesis and also enables us to propose a hypothesis about the minimality of the mean value of the kinetic energy and the minimality of the mean value of the total energy $T - U$ of mechanical systems in stable isolated synchronous motions.

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# ON OSCILLATIONS OF A GYROSTAT AROUND STABLE PERMANENT ROTATIONS*

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The oscillations of a gyrostat with constant gyrostatic moment are investigated by the method of averaging for the Euler, Lagrange and Kovalevskaya cases, which are analogous to the oscillations studied earlier /1/ of a solid around its stable permanent rotations.

We consider the perturbed motion of a gyrostat in the neighbourhood of permanent rotations in a central Newtonian field /2/ whose force function is given by

$$U = -mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) - {}^1/_2\mu(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),\ \ \mu = 3g/R.$$

Here $A, B, C$ are the principal moments of gyrostat inertia, $\gamma_i$ are the direction cosines of the $z$-coordinate axis in the principal axes of inertia, $x_0, y_0, z_0$ are coordinates of the centre of gyrostat mass in the axes of inertia, $m$ is the gyrostat mass, and $g$ is the acceleration due to gravity at a distance $R$ from the gravitating centre.

In the Euler, Lagrange, and Kovalevskaya cases the characteristic equation of the first approximation has one or two zero roots and one or two pairs of pure imaginary roots. Consequently, the transformation from the variables $x_i$ to the amplitudes $a_k, \xi_k$ and phases $u_k$ in matrix form will be

$$x = \sum a_i [\operatorname{Re} V_1(i\omega_i)\cos u_i - \operatorname{Im} V_1(i\omega_i)\sin u_i] + V_2(0)\xi_i \tag{1}$$

where $V_1, V_2$ are non-zero columns for the pure imaginary and zero roots of the associated matrix

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of the first approximation equations.

In the Euler case $(x_0 = y_0 = z_0 = 0)$ the equations of motion of a gyrostat with constant gyrostatic moment $(k_i = \text{const})$ in a homogeneous gravitational field $(\mu = 0)$ allow of the particular solution $p = q = 0;\ r = \omega = \text{const}$ for $k_1 = k_2 = 0,\ k_3 = k = \text{const}$, where $p, q, r$ are the components of the angular velocity along the principal axes of inertia. Setting $p = x_1,\ q = x_2,\ r = \omega + x_3$ in the perturbed motion, we have the equations

$$x_1^{\cdot} + \Omega_1 x_2 + a x_2 x_3 = 0, \quad x_2^{\cdot} - \Omega_2 x_1 - b x_1 x_3 = 0, \quad x_3^{\cdot} - c x_1 x_2 = 0 \tag{2}$$
$$a = (C - B)/A, \quad b = (C - A)/B, \quad c = (A - B)/C$$
$$\varkappa_1 = k/A, \quad \varkappa_2 = k/B, \quad \Omega_1 = a\omega + \varkappa_1, \quad \Omega_2 = b\omega + \varkappa_2$$

We linarize the perturbed motion equations (2) and form the matrix ($D = d/dt$ is the differentiation operator)

$$f(D) = \begin{Vmatrix} D & \Omega_1 & 0 \\ -\Omega_2 & D & 0 \\ 0 & 0 & D \end{Vmatrix}$$

When the conditions /3/ $A > C_1,\ B > C_1$, are satisfied, where $C_1 = C + k/\omega$, gyrostat rotation around the $z$-coordinate axis is stable and the characteristic equation $\lambda\,[\lambda^2 + \Omega_1\Omega_2] = 0$ has the zero root $\lambda_1 = 0$ and the pair of pure imaginary roots $\lambda_{2,3} = \pm i\Omega = \pm i\Omega_1\Omega_2$. If the quantity $C_1 > 0$, then it can be treated as a reduced moment of inertia of the gyrostat for the permanent $z$ axis. In the case when $k$ and $\omega$ are opossite in sign, the quantity $C$ can be negative, hence the motion will always be stable.

The columns of the associated matrix of the pure imaginary roots $\lambda_{2,3} = \pm i\Omega$ are mutually proportional; consequently, by selecting

$$V_1(i\omega) = \begin{Vmatrix} -\Omega_1^{1/2} \\ i\Omega_2^{1/2} \\ 0 \end{Vmatrix}, \qquad V_2(0) = \begin{Vmatrix} 0 \\ 0 \\ 1 \end{Vmatrix}$$

as independent, we write the equations of perturbed motion of the gyrostat in normal coordinates as follows:

$$a_1^{\cdot} = (b\varkappa_1 - a\varkappa_2)\,\Omega^{-1}\xi a_1 \sin u \cos u, \quad \xi^{\cdot} = c\Omega\Omega_2\omega^{-2}a_1^2 \sin u \cos u \tag{3}$$
$$u^{\cdot} = \Omega + \Omega^{-1}\xi\,(ab\omega + a\varkappa_2 \sin^2 u + b\varkappa_1 \cos^2 u)$$

Equations (3) averaged with respect to the angular variable $u$ have the solution

$$a_1 = a_{10}, \quad \xi = \xi_0 \tag{4}$$
$$u = [\Omega + {}^1/{}_2\Omega^{-1}\xi_0\,(2ab\omega + a\varkappa_2 + b\varkappa_1)]\,t + u_0 = \omega^* t + u_0$$

The amplitudes $a_1$ and $\xi$ are considered to be constant in the calculation of the means. The solution (4) obtained for the approximate equations is a solution of the exact equations (not the averaged ones) for $c = 0$, i.e., for $A = B$. The solution (4) describes the gyrostat oscillation in the variables $x_1$ and $x_2$ with period $T = 2\pi/\omega^*$.

The phase trajectories in the space of the variables $x_1, x_2, x_3$ are ellipses whose planes are parallel to the $x_1x_2$ plane. The results obtained for $k = 0$ agree with the results in /1/.

For gyrostat motion in a central Newtonian force field the Euler-Poisson equations allow of the particular solution $p = q = 0,\ r = \omega,\ \gamma_1 = \gamma_2 = 0,\ \gamma_3 = 1$ for $k_1 = k_2 = 0,\ k_3 = k = \text{const}$. Setting $p = x_1,\ q = x_2,\ r = \omega + x_3,\ \gamma_1 = y_1,\ \gamma_2 = y_2,\ \gamma_3 = 1 + y_3$ in the perturbed motion, we arrive at the equations

$$x_1^{\cdot} + \Omega_1 x_2 - a\mu y_2 + a\,(x_2 x_3 - \mu y_2 y_3) = 0 \tag{5}$$
$$x_2^{\cdot} - \Omega_2 x_1 + b\mu y_1 - b\,(x_1 x_3 - \mu y_1 y_3) = 0$$
$$x_3^{\cdot} - c\,(x_1 x_2 - \mu y_1 y_2) = 0, \quad y_1^{\cdot} + x_2 - \omega y_2 + x_2 y_3 - x_3 y_2 = 0$$
$$y_2^{\cdot} - x_1 + \omega y_1 - x_1 y_3 + x_3 y_1 = 0, \quad y_3^{\cdot} + x_1 y_2 - x_2 y_1 = 0$$

The characteristic equation of the linearized system (5)

$$\lambda^2\,[\lambda^4 + m\lambda^2 + n] = 0; \quad m = \omega^2 + \Omega_1\Omega_2 - \mu\,(a + b)$$
$$n = \Omega_1\Omega_2\omega^2 - \mu\,(2ab\omega + a\varkappa_2 + b\varkappa_1)\,\omega + ab\mu^2$$

has two zero roots and two pairs of purely imaginary roots $\pm i\omega_k$.

The perturbed motion equations (5) in normal coordinates take the following form after averaging over the angular variables $u_k$:

$$a_k^{\cdot} = 0, \qquad \xi_k^{\cdot} = 0, \qquad u_k^{\cdot} = \omega_k + (-1)^{k-1} 2^{-1} \sum_{i=1}^{2} (\alpha_{ik}^{(k)} - \beta_{ik}^{(k)})\,\xi_{i0} = \omega_k^* \tag{6}$$
$$\alpha_{ik}^{(k)} = (-1)^{i-1} n\,(\mu^{i-1} a d_{2,3-k} c_{ik} + d_{1,3-k} c_{3-i,k})\,(d_{12} d_{21} - d_{11} d_{22})^{-1}$$
$$\beta_{ik}^{(k)} = (-1)^{i-1} n\,(\mu^{i-1} b d_{ik} c_{2,3-k} + c_{1,3-k} d_{3-i\,k})\,(c_{11} c_{22} - c_{12} c_{21})^{-1}$$
$$c_{1k} = \omega_k^2\,[(\omega^2 - \omega_k^2)\,\Omega_2 - \mu b\dot{\omega}], \qquad c_{2k} = \omega_k^2\,[\omega\Omega_2 - \omega_k^2 - \mu b]$$
$$d_{1k} = \omega_k^3\,(\omega^2 - \omega_k^2 - \mu b), \qquad d_{2k} = \omega_k^3\,(\omega - \Omega_2) \qquad (k = 1, 2)$$

The solution of these equations $a_k = a_{k0},\ \xi_k = \xi_{k0},\ u_k = \omega_k^* t + u_{k0}$ shows that the perturbations $x_i$ and $y_i$ are quasiperiodic functions of time with period $T_k = 2\pi/\omega_k^*$.

In the Lagrange case ($x_0 = y_0 = 0,\ A = B$), the Euler-Poisson equations for a gyrostat with constant gyrostatic moment in a central Newtonian field allow of the particular solution $p = q = 0,\ \gamma_1 = \gamma_2 = 0,\ r = \omega,\ \gamma_3 = 1$ for $k_1 = k_2 = 0$ and $k_3 = k = \text{const}$. Setting $p = x_1,\ q = x_2,\ \gamma_1 = y_1,\ \gamma_2 = y_2,\ \gamma_3 = 1 + y_3$ in the perturbed motion, we obtain the perturbed motion equations

$$x_1^{\cdot} - ax_2 - (b - \mu_1)\, y_2 + \mu_1 y_2 y_3 = 0, \quad x_2^{\cdot} + ax_1 + (b - \mu_1)\, y_1 - \mu_1 y_1 y_3 = 0 \tag{7}$$

$$y_1^{\cdot} + x_2 - ry_2 + x_2 y_3 = 0, \quad y_2^{\cdot} - x_1 + ry_1 - x_1 y_3 = 0$$

$$y_3^{\cdot} - x_2 y_1 + x_1 y_2 = 0, \quad r^{\cdot} \equiv 0$$

$$a = [(A - C)\, r - k]/A, \quad b = mgz_0/A, \quad \mu_1 = (A - C)\, \mu/A$$

The characteristic equation

$$\lambda\, [\lambda^4 + (r^2 - 2b + a^2 + 2\mu_1)\, \lambda^2 + (ar + b - \mu_1)^2] = 0$$

of the linearized system has one zero root and two pairs of purely imaginary roots $\pm i\omega_k$. The transformation of (1) to three amplitudes $a_1, a_2, \xi$ and two phases $u_1$ and $u_2$ has the form

$$x_1 = -\sum_{i=1}^{2} d_{1i} a_i \cos u_i, \quad y_1 = -\sum_{i=1}^{2} d_{2i} a_i \cos u_i \tag{8}$$

$$x_2 = \sum_{i=1}^{2} c_{1i} a_i \sin u_i, \quad y_2 = \sum_{i=1}^{2} c_{2i} a_i \sin u_i, \quad y_3 = \xi$$

$$c_{1k} = r\,(ar + b - \mu_1) - a\omega_k^2, \quad c_{2k} = ar + b - \mu_1 + \omega_k^2$$

$$d_{1k} = -\omega_k\, (\omega_k^2 + b - \mu_1 - r^2), \quad d_{2k} = (a + r)\, \omega_k \quad (k = 1, 2)$$

Substituting (8) into (7), we obtain equations in normal coordinates which will have the following form after the average has been taken over the angular variables $u_k$:

$$a_k^{\cdot} = 0, \quad \xi^{\cdot} = 0, \quad u_k^{\cdot} = \omega_k + (-1)^{k-1} \xi_0\, (2d)^{-1} (\alpha_{kk} + \beta_{kk}) = \omega_k^* \tag{9}$$

$$d = (ar + b - \mu_1)(a + r)\, (\omega_2^2 - \omega_1^2)$$

$$\alpha_{kk} = c_{1,\,3-k} d_{1k} - \mu_1 c_{2,\,3-k} d_{2k}$$

$$\beta_{kk} = c_{1k} d_{1,\,3-k} - \mu_1 c_{2k} d_{2,\,3-k}$$

and their solution will be

$$a_k = a_{k0}, \quad \xi = \xi_0, \quad u_k = \omega_k^* t + u_{k0} \tag{10}$$

It can be concluded on the basis of (10) that the variables $x_k, y_k$ are quasiperiodic functions of time with the periods $T_k = 2\pi/\omega_k^*$.

In the case of gyrostat motion in a homogeneous gravity field, the perturbed motion equations, the averaged equations, and their solutions are obtained from (7)–(10) by substituting $\mu = 0$.

We now examine gyrostat oscillation in a central Newtonian force field in the Kovalevskaya case ($y_0 = z_0 = 0,\ A = B = 2C$). The equations of motion allow of the particular solution $p = \omega,\ q = r = \gamma_2 = \gamma_3 = 0,\ \gamma_1 = 1$ for $k_2 = k_3 = 0,\ k_1 = k = \text{const}$. The perturbed motion equations will be

$$2x_1^{\cdot} - x_2 x_3 + \mu y_2 y_3 = 0 \tag{11}$$

$$2x_2^{\cdot} + (\omega + \varkappa)\, x_3 - ay_3 + x_1 x_3 - \mu y_3\, (1 + y_1) = 0$$

$$x_3^{\cdot} - \varkappa x_2 + ay_2 = 0, \quad y_1^{\cdot} + x_2 y_3 - x_3 y_2 = 0$$

$$y_2^{\cdot} + x_3 - (\omega + x_1)\, y_3 + x_3 y_1 = 0, \quad y_3^{\cdot} - x_2 + (\omega + x_1)\, y_2 + x_2 y_1 = 0$$

$$a = 2mgx_0/A, \quad \varkappa = 2k/A$$

The characteristic equation of the linearized system (11)

$$2\lambda^2\, [2\lambda^4 + (2\omega^2 - 3a + \varkappa\, (\omega + \varkappa) - \mu)\, \lambda^2 + a\, (a - \omega^2 + \mu) + \varkappa\omega\, (\omega^2 - 2a - \mu + \omega\varkappa)] = 0$$

has two zero roots and two parirs of purely imaginary roots $\pm i\omega_k$ for stable permanent rotations $a < \varkappa\omega$, consequently, transformation of the equations to the normal coordinates $a_k, \xi_k, u_k$ is by the following formulas:

$$x_1 = \xi_1, \quad x_2 = -\sum_{i=1}^{2} d_{1i} a_i \sin u_i, \quad x_3 = \sum_{i=1}^{2} c_{1i} a_i \cos u_i$$

$$y_1 = \xi_2, \quad y_2 = -\sum_{i=1}^{2} d_{2i} a_i \sin u_i, \quad y_3 = \sum_{i=1}^{2} c_{2i} a_i \cos u_i$$

$$c_{1k} = a\omega - \varkappa\, (\omega^2 - \omega_k^2), \quad c_{2k} = a + \omega_k^2 - \varkappa\omega$$

$$d_{1k} = \omega_k\, (a + \omega_k^2 - \omega^2), \quad d_{2k} = -\omega_k\, (\omega - \varkappa) \quad (k = 1, 2)$$

The averaged equations

$$a_k^{\cdot} = 0, \quad \xi_k^{\cdot} = 0, \quad u_k^{\cdot} = \omega_k + (-1)^k\, 2^{-1}\, (\alpha_{kk} + \beta_{kk}) = \omega_k^*$$

$$\alpha_{kk} = c_{1,\,3-k}\, [-\xi_1 d_{2k} + \xi_2 d_{1k}](c_{11} c_{22} - c_{12} c_{21})^{-1}$$

$$\beta_{kk} = [-\xi_1\, (2^{-1} c_{1k} d_{2,\,3-k} + c_{2k} d_{1,\,3-k}) + \xi_2\, (c_{1k} d_{1,\,3-k} + 2^{-1} \mu c_{2k} d_{2,\,3-k})](d_{11} d_{22} - d_{12} d_{21})^{-1}$$

have the solution $a_k = a_{k0},\ \xi_k = \xi_{k0},\ u_k = \omega_k^* t + u_{k0}$, which shows that the variables $x_i, y_i$ are quasiperiodic functions of the time with the periods $T_k = 2\pi/\omega_k^*$.

If the gyrostat motion occurs in a homogeneous garvity field ($\mu = 0$), then the motion in the $x_i, y_i$ variables remains quasiperidic.

The presence of a gyrostatic moment results in a change in the oscillation frequencies in the neighbourhood of stable rotations.

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# ON THE NON-UNIQUENESS. OF NON-LINEAR WAVE SOLUTIONS IN A VISCOUS LAYER*

A.V. BUNOV, E.A. DEMEKHIN, and V.YA. SHKADOV

Solutions of the stationary travelling wave type are considered in draining layers of a viscous fluid. A one-parameter family of waves /1/ is studied that softly branches off into the upper branch of the neutral stability curve of the plane-parallel flow and goes over into a negative soliton (phase velocity $c<3$) as the wave number tends to zero. It is shown that this family is not unique: for small values of the parameter $\delta$ characterizing the mass flow rate, a second and third family of waves branches off from it with half the period. The critical value $\delta=\delta_*$ is found for which the bifurcation points of the second and third families merge, while for $\delta>\delta_*$ they go into the complex plane; a dependence of the wave number on $\delta$ for which the bifurcation occurs is obtained analytically. The properties of the second family of the periodic wave and positive soliton type, for which $c>3$ are studied. The solutions are constructed numerically: the periodic solutions are continued in the parameter from the bifurcation points or from the known solutions by using the method of invariant imbedding; the soliton solutions are constructed by joining the linear asymptotic forms as the values of the longitudinal coordinate tend to infinity.

1. Steady wave motions of a viscous fluid in a plane layer on a vertical surface are described in the long-wave approximation by the equation /2, 3/

$$h^3h''' + \delta[6(q-c)^2 - c^2h^2]h' + [h^3 - q - c(h-1)] = 0 \tag{1.1}$$
$$\delta = 3^{-7/9}5^{-1}\gamma^{-1/3}R^{11/9}, \quad \gamma = \sigma\rho^{-1}\nu^{-4/3}g^{-1/3}$$

Here $h(x)$ is the layer thickness, $q$ is the mean flow rate, $c$ is the phase velocity referred to the mean flow rate velocity of the laminar waveless flow, $\sigma$ is the surface tension, $R$ is Reynolds number calculated from the mean flow rate and the layer thickness corresponding to waveless flow, and $x$ is the longitudinal coordinate.

The conditions for periodic waves

$$h(0) = h\left(\frac{2\pi}{\alpha}\right), \quad h'(0) = h'\left(\frac{2\pi}{\alpha}\right), \quad h''(0) = h''\left(\frac{2\pi}{\alpha}\right), \quad \frac{\alpha}{2\pi}\int_0^{2\pi/\alpha} h\,dx = 1 \tag{1.2}$$

and for solitary waves (solitons)

$$h \to 1, \ h^{(n)} \to 0 \ \text{as} \ x \to \pm\infty \tag{1.3}$$

The trivial solution $h(x) \equiv 1$, $q = 1$ corresponds to a plane-parallel waveless flow. As is shown in /2/, a selfoscillating wave solution branches off softly from the trivial solution at the point $\alpha_0 = \sqrt{15\delta}$. The fundamental properties of these solutions are investigated in /2, 4/.

Introducing the small parameter $\varepsilon$, we obtain the following expansion in the semicircle $\alpha = \alpha_0$

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